410. Proposed by A. H. HOLMES, Brunswick, Me.

Given a focus and two tangents to an ellipse, prove that the locus of the foot of the normal corresponding to either tangent is a straight line.

No solution has been received.

411. Proposed by C. N. SCHMALL, New York City.

ABCD is a rectangle of known sides. *BC* being produced indefinitely, it is required to draw a straight line from *A* cutting *CD* and *BC* in *X* and *Y*, respectively, so that the intercept *XY* may be equal to a given straight line. (Unsolved in *Educational Times*.)

REMARK BY L. S. SHIVELY, Mt. Morris, Ill.

This problem cannot be solved with straight edge and compasses only, for an arbitrary length of the given straight line. To prove this, it will be sufficient to show that for a particular length the solution, with these restrictions, is impossible.

Let the length of the given line be twice the diagonal of the rectangle. Suppose the problem to have been solved. Let O be the midpoint of XY. Draw AC and CO. Then triangles COY and ACO are isosceles and we have

$$\angle \text{CAO} = \angle \text{AOC} = 2 \angle \text{OYC}$$

and

$$\angle \text{ACB} = \angle \text{CAO} + \angle \text{OYC} = 3 \angle \text{OYC}.$$

Since the given rectangle is arbitrary we have here a solution for the trisection of an arbitrary acute angle, which is known to be impossible with straight edge and compasses. Hence the proposed construction is also impossible.

Solutions were received, too late for credit in the December number, of 403, from W. H. Johnson; of 404, from S. Lefschetz; of 406, 407, and 408 from M. A. Muzzy and H. E. Trefethen.

CALCULUS.

330. Proposed by C. N. SCHMALL, New York City.

X is a homogeneous function, in the nth degree, of x, y, z; Y is any function of u, v, w. If $ux = vy = wz = 1 + X$...(1), prove that (by Eurler's Theorem),

$$x\frac{\partial Y}{\partial x} + y\frac{\partial Y}{\partial y} + z\frac{\partial Y}{\partial z} + \frac{n}{x}\frac{\partial Y}{\partial u} + \frac{n}{y}\frac{\partial Y}{\partial v} + \frac{n}{z}\frac{\partial Y}{\partial w} = (n-1)\left(u\frac{\partial Y}{\partial u} + v\frac{\partial Y}{\partial v} + w\frac{\partial Y}{\partial w}\right).$$

SOLUTION BY THE PROPOSER.

From (1) we have

$$\log x + \log u = \log y + \log v = \log z + \log w = \log (1 + X)$$

Differentiating, we have

$$\frac{1}{x} + \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{1+X}\frac{\partial X}{\partial x},$$

or

and similarly,

$$\left.\begin{aligned}\frac{x}{u}\frac{\partial u}{\partial x}&=\frac{x}{1+X}\frac{\partial X}{\partial x}-1\\ \frac{y}{u}\frac{\partial u}{\partial y}&=\frac{y}{1+X}\frac{\partial X}{\partial y}\\ \frac{z}{u}\frac{\partial u}{\partial y}&=\frac{z}{1+X}\frac{\partial X}{\partial z}\end{aligned}\right\} \tag{2}$$

We get similar expressions involving the partial derivatives of v and w, with respect to x, y and z.

Now

$$\begin{aligned}x\frac{\partial Y}{\partial x}+y\frac{\partial Y}{\partial y}+z\frac{\partial Y}{\partial z}+\frac{n}{x}\frac{\partial Y}{\partial u}+\frac{n}{y}\frac{\partial Y}{\partial v}+\frac{n}{z}\frac{\partial Y}{\partial w}&\equiv\frac{\partial Y}{\partial u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)\\ &\quad+\frac{n}{x}\frac{\partial Y}{\partial u}+\frac{n}{y}\frac{\partial Y}{\partial v}+\frac{n}{z}\frac{\partial Y}{\partial w}.\end{aligned} \tag{3}$$

From (1),

$$x=\frac{1+X}{u},\quad y=\frac{1+X}{v},\quad z=\frac{1+X}{w}. \tag{4}$$

Hence, the right member of (3) can be written, in the form,

$$\frac{\partial Y}{\partial u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)+\frac{nu}{1+X}\frac{\partial Y}{\partial u}+\frac{nv}{1+X}\frac{\partial Y}{\partial v}+\frac{nw}{1+X}\frac{\partial Y}{\partial w}. \tag{5}$$

But by adding the three equations in (2) we have

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=\frac{x}{1+X}\frac{\partial X}{\partial x}+\frac{y}{1+X}\frac{\partial X}{\partial y}+\frac{z}{1+X}\frac{\partial X}{\partial z}-1. \tag{6}$$

Now by the aid of (6) the expression (5) can be written,

$$u\frac{\partial Y}{\partial u}\left(\frac{x}{1+X}\frac{\partial X}{\partial x}+\frac{y}{1+X}\frac{\partial X}{\partial y}+\frac{z}{1+X}\frac{\partial X}{\partial z}-1\right)+\frac{n}{1+X}\left(u\frac{\partial Y}{\partial u}+v\frac{\partial Y}{\partial v}+w\frac{\partial Y}{\partial w}\right) \tag{7}$$

$$=u\frac{\partial Y}{\partial u}\left(\frac{nX}{1+X}-1\right)+\frac{n}{1+X}\left(u\frac{\partial Y}{\partial u}+v\frac{\partial Y}{\partial v}+w\frac{\partial Y}{\partial w}\right) \tag{8}$$

(by applying Euler's Theorem to the first parenthesis).

$$=u\frac{\partial Y}{\partial u}\left(\frac{nX-X-1}{1+X}\right)+\frac{n}{1+X}\left(u\frac{\partial Y}{\partial u}+v\frac{\partial Y}{\partial v}+w\frac{\partial Y}{\partial w}\right),$$

$$=\frac{1}{1+X}\left\{u\frac{\partial Y}{\partial u}(nX-X-1)+n\left(u\frac{\partial Y}{\partial u}+v\frac{\partial Y}{\partial v}+w\frac{\partial Y}{\partial w}\right)\right\}.$$

Remark.—The right member of the result I first gave is apparently incorrect.

Also solved by Elmer Schuyler.